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A Property of Equivalence

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It is shown that if K, A, B are nonsingular matrices over a principal ideal ring R such that $K\otimes A$ is equivalent to $K\otimes B$, then A is equivalent to B.

Key words: Elementary divisors; equivalence; Kronecker products.

Let R be a principal ideal ring. We write $A\tilde{E}B$, if A and B are matrices over R which are equivalent (see [1] for a complete discussion of this topic). The Kronecker product of any two matrices A and B will be denoted by $A\otimes B$.

The following result was suggested by a remark made by W. D. Wallis in his survey paper [2]:
THEOREM: *Suppose that* K, A, B *are nonsingular matrices over* R *such that* $K\otimes A\tilde{E}K\otimes B$. *Then* $A\tilde{E}B$.

It is not actually necessary to assume that A and B are nonsingular, but doing so simplifies the exposition.

We first prove the following:
LEMMA: *Suppose that the sets*

$$\{k_i+a_j;\ 1\leq i\leq r,\ 1\leq j\leq s\},\ \{k_i+b_j;\ 1\leq i\leq r,\ 1\leq j\leq s\} \tag{1}$$

are the same, where $k_1, k_2, \ldots, k_r, a_1, a_2, \ldots, a_s, b_1, b_2, \ldots, b_s$ *are positive integers. Then the sets*

$$\{a_j;\ 1\leq j\leq s\},\ \{b_j;\ 1\leq j\leq s\} \tag{2}$$

are the same.

Proof: Since the sets (1) are the same, we must have the polynomial identity

$$\sum_{\substack{1\leq i\leq r\\1\leq j\leq s}} z^{k_i+a_j}=\sum_{\substack{1\leq i\leq r\\1\leq j\leq s}} z^{k_i+b_j}$$

Hence

$$\left\{\sum_{1\leq i\leq r} z^{k_i}\right\}\left\{\sum_{1\leq j\leq s} z^{a_j}\right\}=\left\{\sum_{1\leq i\leq r} z^{k_i}\right\}\left\{\sum_{1\leq j\leq s} z^{b_j}\right\},\qquad \sum_{1\leq j\leq s} z^{a_j}=\sum_{1\leq j\leq s} z^{b_j},$$

which implies that the sets (2) must be the same. This completes the proof.

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We now prove the theorem. Let π be any prime of R. Let

$$\pi^{k_1}, \pi^{k_2}, \ldots, \pi^{k_r}$$

be the elementary divisors of K which are powers of π,

$$\pi^{a_1}, \pi^{a_2}, \ldots, \pi^{a_s}$$

the elementary divisors of A which are powers of π, and

$$\pi^{b_1}, \pi^{b_2}, \ldots, \pi^{b_t}$$

the elementary divisors of B which are powers of π. Then the elementary divisors of $K \otimes A$ which are powers of π are

$$\pi^{k_i+a_j}, \; 1 \leqslant i \leqslant r, \; 1 \leqslant j \leqslant s; \tag{3}$$

and the elementary divisors of $K \otimes B$ which are powers of π are

$$\pi^{k_i+b_j}, \; 1 \leqslant i \leqslant r, \; 1 \leqslant j \leqslant t \tag{4}$$

(see [1], chapter 2).

Since $K \otimes A \tilde{E} K \otimes B$, (3) and (4) must be the same (so that $s=t$) and hence the sets

$$\{k_i + a_j; 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant s\}, \{k_i + b_j; 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant s\}$$

must be the same. By the Lemma, the sets $\{a_j; 1 \leqslant j \leqslant s\}$, $\{b_j; 1 \leqslant j \leqslant s\}$ must also be the same. Hence the sets

$$\{\pi^{a_j}; 1 \leqslant j \leqslant s\}, \{\pi^{b_j}; 1 \leqslant j \leqslant s\}$$

are the same, so that the elementary divisors of A and B which are powers of any fixed prime are the same. It follows that A and B have the same elementary divisors, and hence that $A \tilde{E} B$. This completes the proof.

The same method of proof also shows that if A, B are nonsingular matrices over R such that

$$A \otimes A \otimes \ldots \otimes A \tilde{E} B \otimes B \otimes \ldots \otimes B,$$

where there are the same number of A's as there are B's in the Kronecker products, then $A \tilde{E} B$.

References

[1] Newman, Morris, Integral Matrices (Academic Press, New York, 1972).

[2] Wallis, W. D., On the Number of Inequivalent Hadamard Matrices. Proc. Second Manitoba Conference on Numerical Math., 383–401(1972).

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